APPRoFi PROJECT - PROBABILISTIC METHODS FOR THE RELIABILITY ASSESSMENT OF STRUCTURES SUBJECTED TO FATIGUE

Nicolas Gayton[1*]

M. Afzali[2], A. Bignonnet[3], PA. Boucard[4], G. Defaux[5], JL. Dulong[6], B. Echard[1], A. Ghouali[7], F. Lefebvre[2],

M. Lemaire[1], M. Masmoudi[8], A. Notin[6], D. Néron[4], A. Ostmane[9], A. Pyre[7], N. Relun[4]

1. Clermont Université, Institut Français de Mécanique Avancée, EA 3867 Laboratoire de Mécanique et Ingénieries, BP 10448, 63000 Clermont-Ferrand, France,
Email: nicolas.gayton@ifma.fr

2. CETIM, 3. AB Consulting, 4. ENS Cachan/ LMT, 5. Phimeca Engineering, 6. UTC, 7. Snecma Vernon, 8. MODARRT, 9. Snecma Villaroche,

* Corresponding author

Abstract. Most of the industries use the deterministic methods codified in design rules or standards to design structures subjected to fatigue failure. Fatigue phenomena are highly affected by uncertainties (mainly fatigue strength of materials and loadings) which have to be taken into account in the design process to manage the structural reliability. This was the aim of the DEFFI project [1], ended in 2008, which dealt with the application of a probabilistic "stress-strength" method [2] to various industrial applications (aerospace, aeronautic, railway, …). In 2008, the APPRoFi project, funded by the French National Research Agency (ANR), was launched with industrial and academic partners to give a general point of view and generic methods for the reliability assessment and robust design of structures subjected to fatigue. Advanced probabilistic methods have been investigated for the statistical modelling of random variables with statistical correlation (material properties), fatigue strength (SN curves), uncertain loadings (Rainflow matrix). In addition, advanced methods have been developed for low cost probability computation based on metamodels (Kriging, sparse grid) and mechanical computation accelerations.

1 INTRODUCTION

Structural design against fatigue failure is a key issue for industrials since 80% of structures currently in-service are concerned by fatigue loadings. Today, most of the industrial companies use deterministic rules (codified in standards or habits based on industrial knowledge) with more or less obvious safety factors. Note for example the European standard EN-13796 [3] for transportation vehicles with cable or the FEM1.001 [4] for crane structures. These codes and rules are all based on codified load spectra that must be compared to deterministic conservative fatigue strength of material. It can be called a deterministic "stress-strength" approach. Fatigue phenomena are highly subjected to uncertainties (loadings, material strength, …) and deterministic considerations in design process lead to questions on structural reliability: what is the probability of failure of the designed structures? What is the safety margin involved? How can dimensions be kept to a minimum without jeopardising the reliability of a structure? Which variables must be considered whose fluctuations can have serious repercussions on the reliability of a structure? A certain number of answers can be provided using probabilistic approaches whose upstream scientific developments are gradually finding their way into industrial research.

To take into account uncertainties in fatigue design, a probabilistic approach named the "stress-strength" method was developed [2] to assess and manage the probability of failure of structures subjected to fatigue loadings. This approach is based on experiences and measurements to quantify uncertainties separately on stress (load effect) and strength. More recently, the APPRoFi project, funded by the French National Research Agency (ANR), was launched in 2008 with industrial and academic partners. It aims at investigating in details advanced methods for the consideration of fatigue uncertainties for probabilistic assessment and sensitivity analysis. The basis of the APPRoFi project is the well established probabilistic "stress-strength" method and five main issues have been identified.

Issue #1 - statistical modelling of variables from data: geometrical variables, material variables and loadings were considered. Each of them has special features that have to be considered: small deviation for geometrical variables, statistical correlation for material variables and time-dependent loadings. All these variables affect the "stress" side of the problem.

Issue #2 - statistical modelling of SN curve: all industry involved in structural design against fatigue has a set of fatigue experiment results (*S*) Stress *vs* (*N*) Number of cycles before failure. This issue deals with the statistical modelling of this SN curve since a same level of stress would generate variability on *N*.

Issue #3 – mechanical model acceleration for repeated close runs: failure probability assessment needs a high number of mechanical runs for different but close data sets. The aim of this issue is to use existing results to predict close configurations with a low computational demand.

Issue #4 – efficient methods for probability assessment: in all reliability analysis of industrial cases, one issue is the computation time. Issue #3 deals with the reduction of computation time of one mechanical run. Issue #4 aims at managing and reducing mechanical runs to provide failure probability estimation with high confidence level.

Issue #5 – efficient methods for sensitivity analysis and probability managing: aims at considering the robust design of structures providing sensitivity indicators and methods for managing the design for a target reliability level.

Based on these issues, the APPRoFi project is coordinated in Work Package (WP). Each WP gathers academic and industrial partners together following their area of competences around each issue. The APPRoFi partners are focused on two different industrial cases studies provided by SNECMA (linear case study, high cycle fatigue ; non linear material, low cycle fatigue) in order to address different mechanical behaviours. The aim of this paper is not to present the case study results but general considerations on each issue. To get information on the case studies and more practical information, the reader can refer to two other Fatigue Design 2011 papers [5, 6].

This paper is an overview of the methods investigated in the framework of the APPRoFi projet. To begin it presents the basis of deterministic rules of fatigue design (Section 2). From a general point of view, a synoptic showing the probability computation principles and Issue #1 to Issue #5 connexions are presented Section 3. Then, the aim of each issue is detailed, a short bibliography review is presented and choices are discussed. A short conclusion is presented Section 9.

2 "STRESS – STRENGTH" METHODOLOGY

Deterministic rules with more or less obvious safety factors are commonly used for the design of structures against failure. This section exposes first the principles of two existing standards and the practice basis of the APPRoFi industrial partners SNECMA. In a second subsection, the principles of the probabilistic "stress – strength" are presented.

2.1 Deterministic point of view in existing rules and standards – SNECMA practice

Currently, in most of the standards and recommendations, deterministic design is performed to check the structural strength to fatigue failure. Safety factors and pessimistic design value of variables allow the designer to take into account the variability, by decreasing the structural strength or by increasing the fatigue loadings. This is also the case for aerospace engine components where the geometry, the loading and the material characteristics have fixed values.

Each of them is set to a conservative value. For example, the geometry is the thinnest one when checking the rupture or buckling capability of the structure. The loading is chosen at 99 % probability of not being exceeded while the material characteristics correspond to the mean value minus three times standard deviation when verifying the structural strength of the component. The deterministic justification must take into account safety coefficients in order to ensure a risk of failure of each component with a probability lower than a target value, with respect to the considered failure modes. These coefficients of safety are applied on the loading. They accounts for uncertainties related to the statistical evaluation of the loading and of the material data. Such uncertainties may come from the natural scattering of physical phenomena, from the representativeness of the tests performed to derive the statistical distributions of the quantities of interest and from the approximations in the physical and/or mathematical models used to perform the justification.

2.2 Probabilistic method of the DEFFI project

The STRESS-STRENGTH Interference Analysis illustrated Figure 1 was used in the DEFFI Project [1] as a generic method for the reliability assessment. The principle of the method is to associate a level of reliability R, or an acceptable probability of failure $P_f = 1 - R$, to a given design. In order to master the level of reliability, the knowledge of all possible uses of the product and the knowledge of the fabrication quality are required. These data are defined by the statistical distribution of the loading severity of all the possible in-service loads (mean μ_C and st. deviation σ_C) and by the statistical distribution of the fatigue strength of the fabricated products (mean μ_R and st. deviation σ_R), respectively. The first distribution which contains information concerning the load effect is referred to as STRESS (with index *C* in French for "Contrainte"), and the second distribution is referred to as STRENGTH (with index *R* in French for "Résistance").

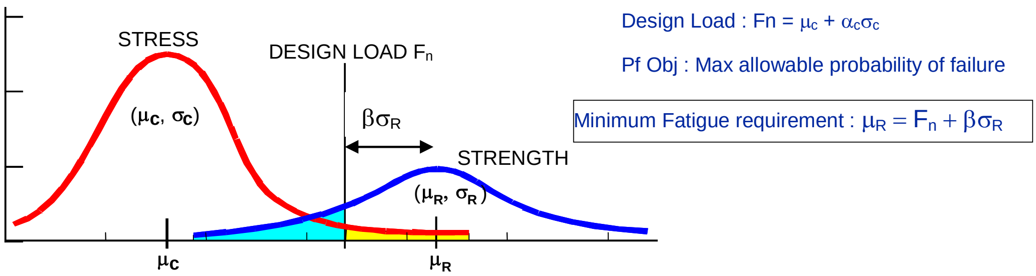

Figure 1- Stress Strength Interference Analysis.

The fatigue loading or STRESS

To quantify the fatigue loading consists in identifying the variability of the in-service loads at each load input point on the component, in each direction. All the essential events leading to fatigue damage must be considered in order to refine the statistical aspects in the scatter of service loading. For most cases the strategy to build a wide range of possible lives of the product is based on targeted measurements and/or simulations of "elementary life situations". To avoid needing too many temporal measurements, the loading can be described as a combination of two types of information:

- the use of the product: the possible proportions of each elementary situation in the whole life of the structure;
- the behavior of the product: the fatigue loads exerted on the structure for each elementary situation.

Elementary situations are characterized by several descriptive ranks. For example in automobile the first rank is given by the type of roads (motorway, road, city, mountain), a second rank is given by the passengers' load (driver alone, half load, full load). In each of these 12 elementary situations several measurements with different drivers are recorded and stored as elementary rainflow matrix standardized to one kilometer. Combinations of these elementary rainflow matrix can be realized allowing the generation of a population of several thousands of possible "virtual lives" represented by their rainfow matrix. For each of these lives an equivalent damage loading representative of the fatigue severity can therefore be determined. From this statistical distribution a severe but representative Design Load F_n with a weak probability of occurrence can be defined (1/100, 1/1000, 1/50000 are current examples).

The Fatigue STRENGTH scatter

The designer should identify the parameters of the probability density function of the strength. But the designer who works with virtual design and a few numbers of prototypes has usually few information on these parameters which would require a production survey. In order to solve this paradox, the coefficient of variation or relative scatter, $q = \sigma_R / \mu_R$, is introduced. This parameter is a stable statistical characteristic of a component family in a controlled fabrication process, it represents the manufacturing quality.

The quantification of the reliability

Let's take z as the damage induces by a given service loading (represented by a load cumulative or its Equivalent Damage Fatigue Loading), the density of probability induces by all the possible in-service loads is $f_C(z)$ and the cumulative distribution function (c.d.f.) of the fatigue components strength is $F_R(z)$ which is function of $\mu_R, \sigma_R = q\mu_R$. The probability of failure before the expected life is given by:

$$\mathrm{P}_f(\mu_R, q) = \int_{\mathbb{R}^+} F_R(z, \mu_R, q) f_C(z) \mathrm{d}z \le \mathrm{P}_f^{\mathrm{target}}$$

If the reliability objective is given by $\mathrm{P}_f^{\mathrm{target}}$, this brings a relation between the mean value (any other fractile can be chosen) of the required fatigue strength μ_R, and the manufacturing quality q.

This approach is particularly well adapted in an industrial context for pre-design in the field of high cycle fatigue with linear mechanical behavior. This can be ballpark or more difficult to use for non-linear mechanical behavior and variability of the loading severity that cannot be modeled with classical standard distributions. In addition, the approach gathers geometrical uncertainties and material uncertainties in the strength side of the problem and the part of each is not quantified. Those are the reasons of a more general approach dedicated to probability evaluation and uncertainties separation.

3 APPROFI ORGANISATION AND GENERAL PROBABILITY COMPUTATION SCHEME

3.1 Failure probability formulation

The aim is to compute the probability P_f that the structure fails before a certain number of fatigue cycle. The formulation of the probability can take several expressions dealing with different performance functions $G(\omega)$. The first one is a formulation in structural damage:

$$P_f = \mathrm{Prob}(G = e(\omega) - 1 \le 0)$$

e is the uncertain (function of the randomness ω) structural damage after its lifecycle loadings. The second one is a formulation in number of cycles:

$$P_f = \mathrm{Prob}(G = N(\omega) - N_{\mathrm{target}} \le 0)$$

N is the uncertain number of loading cycles before failure. N_{target} is the target number of cycles. The third formulation is related to stress as in the probabilistic "stress – strength" approach:

$$P_f = \mathrm{Prob}(G = R(\omega_1) - S(\omega_2) \le 0)$$

The structural strength R and the Stress S are uncertain functions of randomness ω_1 and ω_2. These three formulations have been compared following its ability to consider various type of problems (linear elastic, non linear, …) with few hypotheses and the computation time of the probability. The formulation in damage is the most general one only needing the relation between the time dependent loadings and the damage. This formulation allows taking into account non linear mechanical behavior but is very difficult to take into account in a reliability analysis due to high computation cost. The formulation in number of cycles and stress needs the definition of a number of reference cycles and the use of the damage equivalence principles [7] with the necessary assumption of the mechanical behavior. It has been shown in the project, based on the SNECMA case studies that:

- The formulations in number of cycles and stress are equivalent whatever the mechanical behavior;
- The three formulations are equivalent for linear case studies. For non linear cases studies, formulation in number of cycles and stress brings a scatter in comparison with the formulation in damage. This scatter seems reasonable in the context of the case studies proposed by SNECMA.
- Formulations in stress or in number of cycles are easier to be taken into account in probability and sensitivity analysis.

The formulation in stress is chosen in the APPRoFi framework.

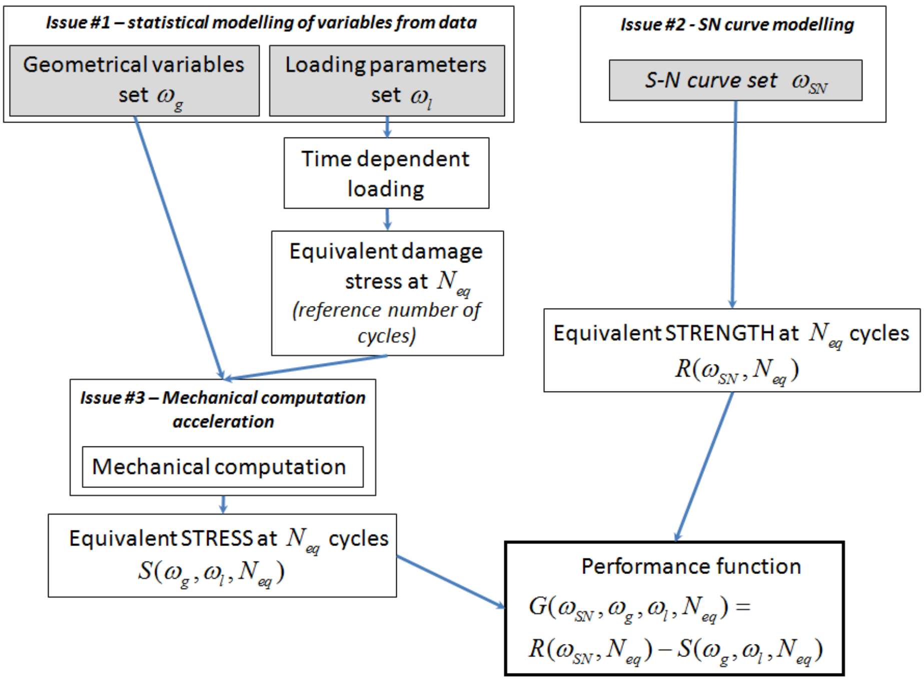

Figure 2 – general computation scheme of the performance function expressed in stress.

3.2 General computational scheme in stress

The general computation scheme is presented Figure 2. It starts from a random sampling of random variables: ω_g for geometry, ω_l for loading parameters and ω_{SN} for the SN curve and the result is the evaluation of the performance function G. This function is positive if the random sampling leads to a safe structure and is negative if the structure falls in the failure domain. The performance function calculation requires fatigue tools such as rainflow counting procedure and equivalent fatigue stress. This computation chain has been implemented with Scilab 5.2 and the mechanical computation has been remote controlled with a dynamic dialogue between partners. Figure 2 does not mention Issues #4 and #5 since they concern the economic way to run this general scheme in order to get the failure probability P_f (Issue #4) and information on the robust design (Issue #5).

4 ISSUE #1 – STATISTICAL MODELLING OF VARIABLES FROM DATA

4.1 Geometrical variables

In the framework of the APPROFI project and of the SNECMA case studies, it has been shown that geometrical variability has no influence on the mechanical response in the frame of their manufacturing tolerances. Consequently, geometrical variability is not studied even if statistical inference method (paragraph 4.2) can be used.

4.2 Material variables modelling

As far material discrepancy is concerned, two different kinds of methods are used to propose a corresponding stochastic model:

- If large data sets are available, classical inference methods are used (frequentist statistics), based on unbiased estimators defined to have expected value equal to the parameter being estimated. A significance test of some null hypothesis and confidence intervals are used to validate the adequation of the identified model to the data. Numerical techniques such as Maximum Likelihood estimates and goodness of fit tests are used to determine the "best" model to introduce in the reliability problem. In particular, such methods are used to identify the material properties.
- If data sets are not sufficient, Bayesian methods are used. A Bayesian analysis synthesizes two sources of information about the unknown parameters of interest. The first of these is the sample data, expressed formally by the likelihood function. The second is the prior distribution, which represents additional (external) information that is available. In particular, it has been used to identify stochastic models dealing with the coefficients of the Chaboche viscoplastic Law for the SNECMA second case study.

4.3 Loading

The first important information to give is that APPRoFi projet focuses only on crack initiation. The structure is considered as failing when the crack initiates. The loading statistical characterisation needs several time measurements of the time dependant loading noted F. It can be a force, a strain or a displacement measurement. The time measurements of $F(t)$ lead to a matrix F_{ij} where i is the measure number and j is related to the moment of the measure. The aim of the loading statistical characterization is to build a stochastic time dependant representation of F. This is a time dependant random field $F(t,\omega_l)$ depending on time and randomness ω_l. In the literature, main references can be found to characterize stationary Gaussian process. The first remark is that measures cannot represent a stationary process since several service conditions are in general measured. The second remark is that, under the assumption of damage independence to fatigue cycles orders, it is only necessary to characterize random rainflow matrix from measurements.

The idea of the DEFFI project is to cut the time measurement regarding characteristic use situations, extract deterministic rainflow matrixes and to mix them with random percentages of each use situation. This idea brings a good representativity of the time dependent loading variations but involves random variables to describe the probable lives of the structure in the reliability analysis. Two ideas are implemented in the APPRoFi project:

- The first one is the random characterization of the cumulative density distribution of equivalent load amplitude at $R=-1$ (SNECMA second case study);
- The second one is to use Nagode work [8] for the characterization of the randomness of the Rainflow matrix.

These two methods are interesting because they involve only one random parameter for the first proposition and at most five random parameters for the second.

One case study of the project has not any loading measurements. Discussion with experts allows us to bring a statistical modelling of loadings.

5 ISSUE #2 – STATISTICAL MODELLING OF SN CURVE

As presented above, the generic reliability design approach requires the identification of the statistical distribution of strength. The quality and relevance of this identification requires sufficient data. This analysis is based on well known fatigue behaviour laws (Basquin, Bastenaire or other) and probabilistic conventional methods. Whilst it is difficult to use traditional method being given a low number of results available, strategies of capitalisation of data needs to be developed. In the APPRoFi project, probabilistic approaches of SN curves are proposed considering a deterministic approach of the mean curve $S_{50\%}$. Four models are presented Figure 3 and a short description is given below:

ESOPE model (AFNOR A03-405) [9]

- Basquin deterministic model: $S_{50\%} = C\,N^{m}$
- Probabilistic approach: S at given N is assumed normally Gaussian distributed with a constant standard deviation σ_S for any N.
- The SN design curve or probabilistic S-N curve is generated by a standard normal random variable u which may be written as:

$$u = \frac{S - S_{50\%}}{\sigma_S} \quad \rightarrow \quad S = S_{50\%} + u\,\sigma_S$$

ESOPE model in log (called « Bignonnet ») [2]:

- Basquin deterministic model in log: $\left[\log S\right]_{50\%} = A \log N + B$
- Probabilistic approach: $\log S$ at a given N is assumed normally Gaussian distributed with a constant standard deviation $\sigma_{\log S}$ for any N.
- The SN design curve is generated by a standard normal random variable u which may be written as

$$u = \frac{\log S - \left[\log S\right]_{50\%}}{\sigma_{\log S}} \quad \rightarrow \quad \log S = \left[\log S\right]_{50\%} + u\,\sigma_{\log S}$$

ESOPE model in log with constant relative scatter (named ESOPE 2):

- Basquin deterministic model in log: $\left[\log S\right]_{50\%} = A \log N + B$
- Probabilistic approach: log(S) at a given N is assumed normally Gaussian distributed with a constant relative scatter $cv_{\log S}$ for any N.
- The SN design curve is generated by a standard normal random variable u which may be written as:

$$u = \frac{\log S - \left[\log S\right]_{50\%}}{cv_{\log S}\left[\log S\right]_{50\%}} \quad \rightarrow \quad \log S = \left[\log S\right]_{50\%} + u\,cv_{\log S}\left[\log S\right]_{50\%}$$

Guédé Model [10,11]: modelling of the random number of cycles knowing S

- Basquin deterministic model in natural logarithm: $\left[\ln S\right]_{50\%} = E \ln N + F$
- Probabilistic approach: $\ln N$ at a given S is assumed normally Gaussian distributed with a constant relative scatter $cv_{\ln N}$ for any N.
- The SN design curve is generated by a standard normal random variable u which may be written as:

$$u = \frac{\ln N - \left[\ln N\right]_{50\%}}{cv_{\ln N}\left[\ln N\right]_{50\%}} \quad \rightarrow \quad \ln N = \left[\ln N\right]_{50\%} + u\,cv_{\ln N}\left[\ln N\right]_{50\%}$$

Different methods of parameter identification of models are considered:

- Estimate the parameters of the deterministic model (median curve) by least squares and then estimate the statistical distribution parameters of the probabilistic model by characterizing the statistic of u.
- Estimate simultaneously model parameters of deterministic and probabilistic model by maximum likelihood techniques [12]. The maximum likelihood estimate is more comprehensive for the determination of all parameters simultaneously, but requires the use of method for optimizing the likelihood $L(x_i, \theta_i)$ function with complex type local minima. x_i are here the data and θ_i the model parameters.

An identification method is proposed in two stages:

- Search the parameters of the deterministic model by least squares (LS) and then estimate the standard deviation or the coefficient of variation. The optimum is then defined.
- Search the optimum in the sense of maximum likelihood using as a starting point the LS optimum. An optimization method is then used to find the optimum by avoiding local minima.

In order to compare the four statistical model, the AIC (Akaike Information Criterion) and BIC (Sawa's Bayesian Information Criterion) criterions are used. Generally speaking, AIC and BIC criterion quantify the quality of a statistical model based on k parameters (here $k=3$ for the four proposed models) from a sample of size n. They are based on the value of the maximum likelihood function $L(x_i, \theta_i^*) = L_{\max}$:

$$AIC = -2\log L_{\max} + 2k$$
$$BIC = -2\log L_{\max} + k\log(n)$$

The minimum values of AIC and BIC are needed to provide the best fitted model.

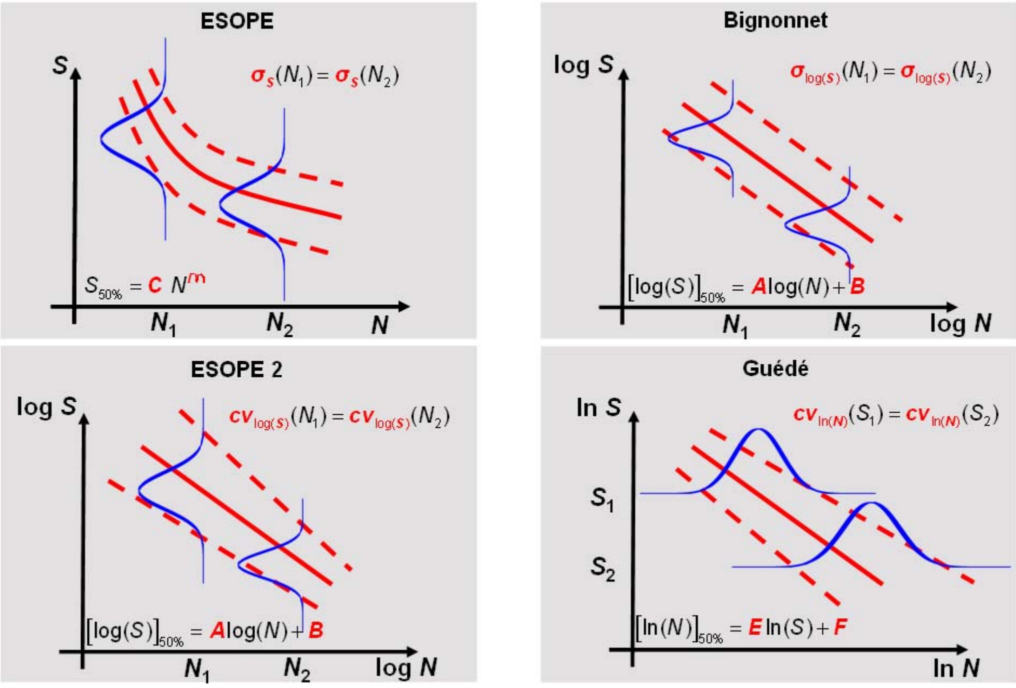

Figure 3- Models for the mean S-N curve. 5% and 95% percentile values in dotted lines.

6 ISSUE #3 – MECHANICAL MODEL ACCELERATION FOR REPEATED CLOSE RUNS

Most of the methods which allow to study failure probability need to perform numerous runs of the same deterministic mechanical problem but for different values of a set of parameters (material properties, boundary conditions, loadings …). Indeed, the issue is the proposal of dedicated techniques to speed up the calculation of these repeated close problems. Two methods are investigated by APPRoFi partners.

6.1 SLDLt method [13]

To speed up the different runs needed to compute the mechanical stress field, a completely new approach, called SLDL^T, is proposed. For first application, this method is restricted to model a random field over a structure. The mechanical problem is linear and the Young modulus is modelled by a continuous stochastic field $\mathbf{E}(x,\omega)$, characterized by a correlation kernel. This field is discretized over the finite element model in a random vector $\mathbf{E}(\omega)$, characterized by a covariance matrix Σ_E and a mean vector $\mathbf{E}_0$. This vector is computed from a non correlated standard log-normal distribution $\eta(\omega)$ as:

$$\mathbf{E}(\omega) = \Sigma_E^{1/2}\,\eta(\omega) + \mathbf{E}_0$$

In static, the displacement vector $\mathbf{u}(\omega)$ is computed by solving the equilibrium equation :

$$\mathbf{K}(\mathbf{E}(\omega))\,\mathbf{u}(\mathbf{E}(\omega)) = \mathbf{f}$$

Since $\mathbf{K}$ is a symmetric positive definite matrix, this equation is usually solved using the LDL^T decomposition of $\mathbf{K}$:

$$\mathbf{K}(\mathbf{E}(\omega)) = \mathbf{L}(\mathbf{E}(\omega))\, \mathbf{D}(\mathbf{E}(\omega))\, \mathbf{L}(\mathbf{E}(\omega))^T$$

For each sample ω, a new decomposition is required: it is the most time-consuming step in the computation. The idea developed in our new SLDL^T method is to store the LDL^T decomposition of a reference sample ω_0, and to keep its matrix $\mathbf{L}_0 = \mathbf{L}(\mathbf{E}(\omega_0))$ over all samples ω. Hence, the exact stiffness matrix $\mathbf{K}(\mathbf{E}(\omega))$ is approximated by :

$$\mathbf{K}^*(\mathbf{E}(\omega)) = \mathbf{L}_0\, \mathbf{D}(\mathbf{E}(\omega))\, \mathbf{L}_0{}^T$$

Of course, this hypothesis introduces an error which is corrected, partially, by adjusting the diagonal matrix $\mathbf{D}(\mathbf{E}(\omega))$ to minimize the following error, defined by the Frobenius norm :

$$J_E(\mathbf{d}) = ½\, \| \mathbf{K}(\mathbf{E}(\omega)) - \mathbf{L}_0\, \mathbf{D}(\mathbf{E}(\omega))\, \mathbf{L}_0{}^T \|_F$$

The vector $\mathbf{d}$, defined as the diagonal vector of the diagonal matrix $\mathbf{D}$, is then the solution of the following linear system:

$$\mathbf{A}\, \mathbf{d}(\mathbf{E}(\omega)) = \mathbf{b}(\mathbf{E}(\omega))$$

where:

$$(\mathbf{A})_{ij} = [\, (\mathbf{L}_0{}^T)_{j.}\, (\mathbf{L}_0)_{.i}\,]^2$$

$$(\mathbf{b})_i\, (\mathbf{E}(\omega)) = (\mathbf{L}_0{}^T)_{i.}\, \mathbf{K}(\mathbf{E}(\omega))\, (\mathbf{L}_0)_{.i}$$

For computing purpose, the matrix $\mathbf{A}$ is independent of the sample ω, and so can be inverted once. Finally, the time to compute $\mathbf{d}$ is governed by the time to compute the right hand side $\mathbf{b}$. Now, restricting this method to the case of a stochastic Young modulus field, the computation of $\mathbf{b}$ can be drastically reduced. Indeed, for each finite element e, the associated Young modulus $E^e(\omega)$ can be factorized from the elementary stiffness matrix $\mathbf{K}^e$, and so:

$$\mathbf{K}(\mathbf{E}(\omega)) = \textstyle\sum_e E^e(\omega)\, \mathbf{K}^e$$

After calculation, $\mathbf{b}$ writes:

$$\mathbf{b}(\mathbf{E}(\omega)) = \mathbf{B}\, \mathbf{E}(\omega)$$

The matrix $\mathbf{B}$ is also independent of the sample θ. Finally, the diagonal $\mathbf{d}$ writes:

$$\mathbf{d}(\mathbf{E}(\omega)) = \mathbf{A}^{-1}\, \mathbf{B}\, \boldsymbol{\Sigma}_E^{1/2}\, \boldsymbol{\eta}(\omega) + \mathbf{A}^{-1}\, \mathbf{B}\, \mathbf{E}_0 = \mathbf{C}\, \boldsymbol{\eta}(\omega) + \mathbf{c}$$

The matrix $\mathbf{C}$ and the vector $\mathbf{c}$ can be computed once. And so, the time to compute the displacement $\mathbf{u}(\mathbf{E}(\omega))$ is reduced to the time of forward-backward substitution of $\mathbf{L}_0$. The error induced by the approximation can be controlled by an error estimator, not developed here. This new approach to solve a large number of random samples, in a Monte-Carlo simulation, is a promising way to model stochastic fields.

For the application in the APPRoFi project, this method permits to evaluate the STRESS side of the reliability problem in a reduced computer time. Several developments are on going to consider non-linear mechanical problems approximating tangent matrices with this SLDLt principles.

6.2 LArge Time Increment (LATIN) method and PGD

The two aims of this work were: (i) to perform each of the deterministic simulations quickly, even if the problem is nonlinear and involves a huge number of degrees of freedom; (ii) to propose a way to reuse already known results to predict close configurations (from one set of parameters to another) at a small computational cost.

The strategy which has been used is based on the LArge Time Increment (LATIN) method [14]. Roughly, the LATIN method is an iterative algorithm, which generates an approximate solution of the problem over the whole time-space domain and improves the quality of this approximation automatically at each iteration, even starting with a very coarse first prediction. Each of the iteration consists of two stages:

- a local stage, during which a solution that satisfies the local-in-space equations is sought, in particular the material nonlinear constitutive relation;

- a linear stage, during which a solution that satisfies the linear equations is sought, in particular the global-in-space equilibrium equations.

This decoupling between the nonlinear and the global equations, as well as the nonincremental characteristic of the algorithm allows to introduce a separated representation of the unknowns. This method, which is called Proper Generalized Decomposition (PGD), is closely related to Proper Orthogonal Decomposition (POD) because it consists in the approximation of a field $f(t, M)$, defined over the time-space domain, by a finite sum decomposition of products of a time function by a space function:

$$f(t,M) \approx \sum_i \lambda_i(t)\Lambda_i(M)$$

However, the huge difference is that the PGD approach does not imply the full knowledge of f to build the decomposition but only the knowledge of the equations that define this field.

Using PGD, the LATIN algorithm consists in building the approximation by successive enrichments along the iterations and up to a prescribed accuracy. At a particular iteration n+1 of the LATIN, functions $\Lambda_i(M)$ ($\forall i \leq n$) are known from the previous iterations and can be seen as a reduced basis which will be first used to seek the new approximation of the solution. The dimension of the reduced basis being small, this computation is nearly inexpensive. If the quality of this approximation is not sufficient, a new functional product is determined in sequence by using a fixed-point alternating-directions strategy between a problem defined only over the time and a problem defined only over the space domain. This is the time consuming part of the algorithm but it remains very cheap compared to the resolution of the entire nonlinear problem at each time step [15].

Concerning the computation of the solution for numerous sets of parameters, a strategy which is called “multiparametric” is used [16]. It is based on the concept of Reduce Order Modelling (ROM). Suppose that the PGD of the solution for a first set of data is known. To compute the solution for a new set, the reduced basis which has been already generated is used to initialize the algorithm, which is quite inexpensive and generally sufficient if the two sets of data are close. An error indicator allows to know if the required quality is achieved and to automatically enrich the approximation by adding new functional products if it is not the case. Some dedicated techniques have been setup to follow the best path in the parametric space when performing a parametric study and the CPU gain obtained for typical engineering problems with 1,000 parameter sets can reach more than 30.

7 ISSUE #4 – EFFICIENT METHODS FOR PROBABILITY ASSESSMENT

In the last decades, structural reliability analysis has gone from academic applications to industrial one dealing with complex time consuming mechanical models mainly thanks to the use of metamodels also called surrogate models or response surfaces. The principles consist in replacing the expensive models by a simple but representative one based on a learning steps with an adaptive (or not) numerical experimental design. Lots of metamodel types have been tested: polynomial approximations, neural network, SVM. More recently kriging metamodels has been investigated in an adaptative scheme to predict the failure probability. The main advantage is the Kriging variance that is a free quality predictor. The second type of metamodels deals with sparse grids that consist in driving the experimental design following the sensitivity of the response to each variable. This kind of methodology is effective for high dimension problems.

7.1 AK-MCS based kriging method [17]

Reflexion on the way to minimize the finite element runs to predict the probability that the performance function is negative leads to the proposition of the AK-MCS method. All the usual metamodels based approach try to approximate the function G. This is efficient for sensitivity analysis but not optimal for reliability analysis since only the sign of G is interesting. This is the basis of the AK-MCS method. Here are the main steps:

1) a Monte Carlo sample with n_{MC} potential numerical experiments is randomly chosen;
2) ten experiments chosen randomly are computed with the FE code;

3) a Kriging metamodel is evaluated from the ten results;
4) based on the prediction and on the kriging variance, the probability $U(x)$ to mispredict the sign of G in all the non computed point of the Monte Carlo sample is evaluated;
5) the point with the highest probability $U(x)$ is computed with the FE code, return to step 3 until the largest probability is lower than a stop value;
6) When the probability stop value is reached, the failure probability is evaluated onto the kriging based metamodels.

AK-MCS methods proved its efficiency on various types of problems, from complex academic analytical tests to industrial one. To reach very small failure probability, that is the case of the SNECMA case studies, a deal between FORM techniques and importance sampling using AK-MCS method classification has been implemented. Details are given in the paper related to the aerospace case study in this conference [28].

7.2 Sparse grid based method

Conventional meta-model construction techniques relies on generating distributed experimental points, prior to the approximation process, without taking into account the specific behaviour of the function to be approximated. These techniques fail when design parameters are large owing to curse of dimensionality [18]. Sparse grids are a well-studied technique first applied in numerical solution of partial differential equations (PDE) to address the curse of dimensionality. More specifically, the meta-modelling problem considered with sparse grid interpolation is an optimal recovery problem that is, the selection of points such that a smooth multivariate function can be approximated with a suitable interpolation basis [19].

The basis of all sparse grid based meta modelling technique is the Smolyak's construction, that is univariate interpolation. Formulae are extended to the multivariate case by using tensor products [18] and hierarchical basis functions. As a result, a powerful interpolation method that requires significantly fewer support nodes than conventional interpolation on a full grid is obtanied. A more detailed description of the sparse grid algorithm, including grid construction, error bounds, etc., is given by [20]. Here we only highlight the accuracy and computational cost of the method. Consider the *d*-dimensional, C^k smooth function $f(X), X \in [-1\ 1]^d$. Let $\tilde{f}(X)$ denotes sparse grid meta-model of function f and N be the total number of grid points used to construct the meta-model. For piecewise linear basis function the interpolation error is $\mathrm{O}\left(\mathrm{N}^{-2}\left|\log \mathrm{N}\right|^{(3(\mathrm{d}-1))}\right)$ whereas for polynomial basis function the interpolation error is $\mathrm{O}\left(\mathrm{N}^{-\mathrm{k}}\left(\log \mathrm{N}\right)^{(\mathrm{k}+2)(\mathrm{d}-1)+1}\right)$ with maximum norm [19,20]. This shows that, the dimension d occurs only in the logarithmic term, indicating the method's excellent suitability for high-dimensional problems. Unlike conventional techniques for which interpolation error is given by $\mathrm{O}\left(\mathrm{N}^{-\mathrm{k}/\mathrm{d}}\right)$, the computational cost for sparse grid based interpolation does not increase exponentially with dimension and thus can handle curse of dimensionality. The computational cost is further reduced by employing dimensional adaptivity [20,21]. The adaptive algorithm begins with a low level grid, and automatically detects directions for which the current approximation error is large. Additional points are only added in these directions (grids are refined) thus resulting in more refinement levels for the significant variables and low levels for the others. This way the dimensional adaptivity of sparse grid method automatically detects most significant model parameters during construction process. Once a sparse grid meta-model is constructed, global sensitivity information can be extracted efficiently [22]. Unlike wavelets with isotropic basis functions at a given level, adaptive sparse grid algorithm results into anisotropic basis functions, much better adapted to the real life problems.

8 ISSUE #5 – EFFICIENT METHODS FOR SENSITIVITY ANALYSIS AND PROBABILITY MANAGING

To manage the structural probability of failure, various kinds of information and methods can be taken into account. First, sensitivity analysis, by the way of numerical gradients, can provide information on the characteristic values that have a significant influence on the failure probability. Moreover, FORM techniques or global sensitivity analysis can give information on the variance importance of each variable on the probability or on the variance of the structural damage. Secondly, reverse problem can be solved to set the failure probability to a target value from only one design parameter. More generally, dealing with more than one design parameter, an RBDO (Reliability Based Design Optimization) problem can be addressed to minimize the production cost subjected to certain level of failure probability. Here are the basis of sensitivity and RBDO analysis.

8.1 Sensitivity analysis

Numerous sensitivity techniques exist following the needed information:

- Local sensitivities study how a small local perturbation of parameters can be transmitted through the mechanical model or on the failure probability. In the framework of reliability problems, FORM algorithm can provide information (squared direction cosines) on the failure probability sensitivity to standard deviation of all the random variable [23].
- Screening techniques [24] consists in a generalization of the local sensitivities. They are qualitative methods because they allow to sort the design variables by their relative influence on the response function.
- Global sensitivity analyses study the impact of each variable on the response variance (Sobol' indices [25]) or the whole statistical distribution of the response (Borgonovo's indices [26]).

In the framework of the APPRoFi project, FORM techniques are applied with importance sampling classification in order to reach low probabilities. As a consequence, squared direction cosines are freely available to quantify the importance of each random variable on the failure probability: that is one of the information interesting for industrial partners.

8.2 Reliability Based Design Optimization

Designs based on a deterministic formulation do not account for uncertainty in design parameters. Such designs are associated with a high probability of failure if the design constraints are active at the deterministic optimal solution. When uncertainties are small and product service life is not very critical, use of safety factor in design can account for the uncertainty in the design parameters. But designs based on safety factor are often over-conservative. They lack the ability to achieve specified level of constraint satisfaction, therefore are risky. The desired level of constraint satisfaction can be expressed as a specified probability of failure to satisfy the constraint. Mathematically the optimal design under uncertainty is a constrained optimization problem in which constraint is on the value of probability of failure. The problem can be formulated as:

$$X^* = \underset{X \in D_s}{\arg\min}\, O_{bj}(X) \text{ such that } D_s = \left\{ X \in \mathbb{R}^n, P_f = \text{Prob}\left(G(X) \le 0\right) \le P_{\text{target}} \right\}$$

$$P_f = \text{Prob}\left(G(X) \le 0\right) = \int_{G(X) \le 0} f_X(X)\,dX$$

where, O_{bj} is the design objective, D_s is the design space spanned by design variables X , P_f is the probability of failure, $G(X)$ is the performance or limit state function, f_X is joint probability density function, P_{target} is the targeted probability of failure for safe design, and X^* is the optimal design point. Depending on the design goals the objective function can be defined in different ways, for example,

- $O_{bj} = F(X)$, the design cost
- $O_{bj} = \mathrm{E}[F(X)]$, the mean value of the design cost if the goal is a robust design
- $O_{bj} = \text{Prob}[F(X) < F^*]$, that is the probability of failure less than specified values if the design is for the extreme conditions.

9 CONCLUSIONS AND EXTENSION TO APLICATIONS

In this paper, the methodologies and methods developed by the APPRoFi project are summarized. The reliability analysis of an in-service structure against fatigue can be verified following the general synoptic Figure 2. This synoptic is composed by bricks corresponding to the issues discussed in the paper. Several strategies can approached for each brick. The whole synoptic schema can be used whatever fatigue problem with most familiar methods for reliability analysis, mechanical evaluation, statistical modelling. The choice of each method depends on:

- the complexity of the problem (size of model, computational time for mechanical analysis, non linear behavior);
- the statistical knowledge and the number of available data.

The APPRoFi project allowed performing the research in the area of stochastic fatigue analysis with industrial space and aerospace applications. Other main achievements in this project could be summarized as follow:

- Extensive study in probabilistic approaches and modelling of input data as loading, material data from the limited existing data. This an essential issue for the industrial applications;
- to develop the appropriate statistical model for the heterogenic existing data;
- to extend the DEFFI project results for stochastically modelling of the material fatigue behavior law;
- to develop the methodologies, the numerical mechanical analysis techniques and to propose the advanced techniques for the model reduction saving the computational time for Finite Element analysis up to 30 comparing with classical techniques. The original methods which have been developed in the project allowed to drastically reduce the computational time necessary to build metamodels. This is one of the main issues for using the stochastic approach for the design of the industrial mechanical components and structures.
- Sensitivity analysis provides interesting information to show the importance of each random variable on the structural reliability.

For details on the industrial applications, the reader can refer to [28, 29] where all allowable for publishing information are provided.

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